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COMPUTING PSEUDOSPECTRA USING BLOCK ARNOLDI ITERATION

REINALDO ASTUDILLO* AND ZENAIDA CASTILLO†

Abstract. The *pseudospectra* is a useful tool to study the behavior of systems associated with nonnormal matrices. Different projection Krylov methods have been used to calculate the *pseudospectra* of large matrices rather than typical aproaches which require the application of SVD descomposition several times, inverse power method or Lanczos method. In this work we investigated the practical applicability and the performance of Block Implicit restarted Arnoldi method (BLIRAM) proposed in [5] to approximate the pseudospectrum of large matrices.

Key words. Pseudospectrum, Implicit restarted block Arnoldi, Block Krylov subspace methods.

1. Introduction:. The spectrum of a matrix $A \in \mathbb{C}^{n\times n}$ is the set of all $\lambda \in \mathbb{C}$ such as $Ax = \lambda x$, for some non-null vector x called eigenvector. This set can be described as:

$$\Lambda(A) = \{z \in C : rank(zI - A) < n\} = \{z \in C : (zI - A) \text{ is singular}\}$$

where $rank(A)$ is the number of columns linearly independent.

Computing eigenvectors and eigenvalues of large matrices is an important step in several applications in sciences and engineering, however when this matrices are highly non-normal ($AA^H \neq A^HA$) and sparses, the information of the spectrum is insufficient to analize some important phenomena, for example, the behavior of solution of dinamical systems or operators with its perturbations [6]. In recent years *pseudospectra* of matrices has become an important tool to study dynamical system with non-normal operators asociated. This concept was introduced (with different names): by Landau (1975) [10], Varah (1967) [21] , Kostin and I. Razzakov (1985) [9], Gudonov (1992) [7], and L. Trefethen in 1990 and 1992 [19]. Given $\epsilon > 0$, the *pseudospectra* of a matrix $A \in \mathbb{C}^{n\times n}$, can be defined in different equivalent ways:

DEFINITION 1.1. *The ϵ-pseudospectra of matrix A denoted by $\Lambda_\epsilon(A)$ is the set of numbers $z \in \mathbb{C}$ such that:*

$$\|(zI - A)^{-1}\| > \epsilon^{-1}, \tag{1.1}$$

when $\| \star \|_2$ is used, the following equivalent definition is obtained:

DEFINITION 1.2. *$\Lambda_\epsilon(A)$ is the set of $z \in \mathbb{C}$ such that:*

$$s_{min}(zI - A) < \epsilon \tag{1.2}$$

where $s_{min}(A)$ denotes the minimun singular value of A. From perturbation eigenvalue theory two more definitions are in order:

DEFINITION 1.3. *$\Lambda_\epsilon(A)$ is the set of $z \in \mathbb{C}$ such that:*

$$z \in \Lambda(A + E)$$

for some $E \in \mathbb{C}^{n\times n}$ with $\|E\| < \epsilon$. $\Lambda(A)$ denote the set of eigenvalues of A.

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DEFINITION 1.4. *$\Lambda_\epsilon(A)$ is the set of $z \in \mathbb{C}$ such that:*

$$\|(zI - A)v\| < \epsilon$$

for some $v \in \mathbb{C}^n$ with $\|v\| = 1$, the vector v is called pseudoeigenvector and z pseudoeigenvalue.

1.1. Notation:. We will generally adopt Householder notation: capital letters $A, B, C, \ldots$ denote matrices, A^T is the transpose of the matrix A, and A^H is its conjugate transpose, lower case letters except z denote column vectors. Greek letters and z represent complex scalars. I_n is the identity matrix, and wherever the context is clear the subindex n is eliminated. $\Lambda(A)$ is the spectrum of the matrix A, while $\Lambda_\epsilon(A)$ is its ϵ-pseudospectrum, the simbol $s_{min}(A)$ denote the smallest singular value of A, and $\lambda_{min}(A)$ represents the smallest eigenvalue in magnitude. The algorithm are presented in Matlab notation and pseudoformal notation.

2. A brief comparison between spectrum and pseudospectrum analysis. To ilustrate the diference between spectrum and pseudospectrum analysis, an example taken from the paper of E. Mengi and M. Overton "Algorithms for the computation of pseudospectro radius and numerical radius of a matrix" [13] is presented.
Suppose we want to predict the behavior of $\|A^k\|$, where $A \in \mathcal{C}^{n\times n}$. When all the eigenvalues of the matrix A lie inside the unitary circle and $k \to \infty$ it is well-known that $\|A^k\| \to 0$, moreover the asymptotic rate of decay of $\|A^k\|$ when $k \to \infty$ can be measured by the spectral radius of the matrix A:

$$\rho(A) = max\{|\lambda| : \lambda \in \Lambda(A)\}$$

However, for finite values of k the speudospectrum itself does not reveal any information about the behavior of $\|A^k\|$, unless the matrix A is normal ($AA^H = A^HA$). When A is a non-normal matrix, $\|A^k\|$ can be arbitrarily large even when the whole spectrum is inside the unitary circle. In this case the pseudospectrum of the matrix A enters as an useful tool, a component of the ϵ-pseudospectrum of the matrix A is the pseudospectral radius:

$$\rho_\epsilon(A) = \max\{|z| : \forall z \in \Lambda_\epsilon(A)\}.$$

with this definition it can be deduced that (see [23, 6]):

$$\sup_{\epsilon>0} \frac{\rho_\epsilon(A) - 1}{\epsilon} \leq \sup_k \|A^k\| \leq e\, n \sup_{\epsilon>0} \frac{\rho_\epsilon(A) - 1}{\epsilon}, \tag{2.1}$$

where $e = \sum_{i=0}^{\infty} 1/i!$. In this way the norm of the powers of A are bounded in terms of the pseudospectral radius. The left-bound of the inequality (2.1) provides information about how large the norms of the matrix power can grow. For example, consider the Toeplitz pentadiagonal matrix A defined by:

$$a_{ij} = \begin{cases} -0.4 & \text{si } j = i+1; \\ 0.4 & \text{si } j \leq i \text{ y } 1 \leq j \geq 5\ , \quad . \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

For this matrix $\rho(A) = 0.9052$, which means that all the spectrum of the matrix A is inside of the unit circle, then $\|A^k\|$ will converge to zero as k converges to infinity,

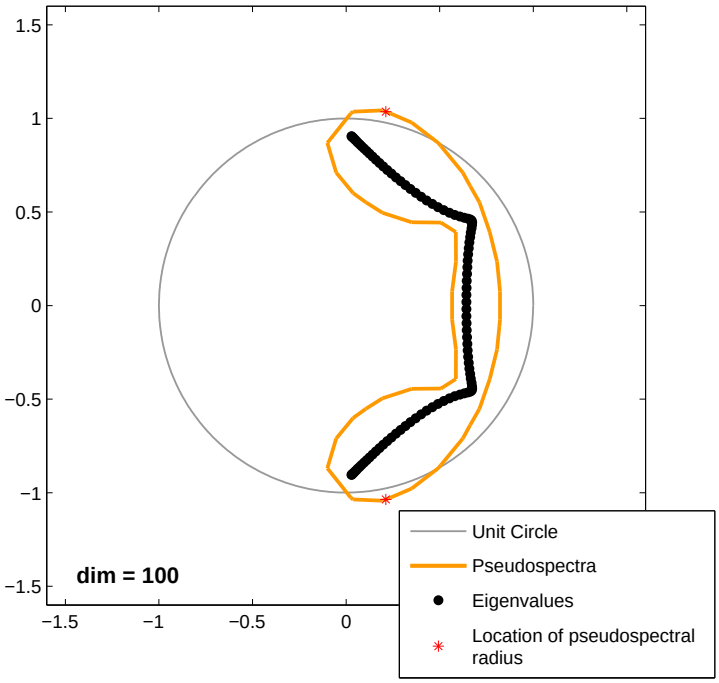

FIG. 2.1. *ϵ-Pseudospectra of the matrix A (2.2) with $\epsilon = 10^{-7}$*

however as we can see in figure 2.1 for $\epsilon = 10^{-7}$ the $\epsilon-$pseudospectrum does not lie inside the unit circle. Using this information and the inequaties (2.1) we can predict that the norms of A^k can be at least 5.72×10^5 for some fixed value of k (see figure 2.2), and the spectral analysis itself cannot reveal any information about this behavior.

3. Pseudospectra. A classical algorithm to compute the *$\epsilon-$pseudospectrum is sumarized in the following steps [4]:*

1. *Determine a region $\mathcal{K}$ of interest in $\mathcal{C}$.*
2. *Discretise the region $\mathcal{K}$.*
3. *For each point z of the discretisation, compute $\|(zI - A)^{-1}\|$.*
4. *Use a visualisation tool to display the computed value for $z \in \mathcal{K}$.*

There are different ways to determine and discretise the region $\mathcal{K}$. For a discussion of this topic see [6, 2]. To calculate $\|(zI-A)^{-1}\|$ we can use different approaches:

3.1. Basic SVD method. *For each point z of the discretisation, we can compute the singular value descomposition of the matrix $(zI - A)$ and select the minimal singular value. A Matlab version of this algorithm is:*

```
for k=1:m
    for j=1:m
        sigmin = min(svd((x(k)+y(j)*1i)*eye(N) - A)); %comnpute svd factorization
    end
end
contour(x,y,log10(sigmin));
```

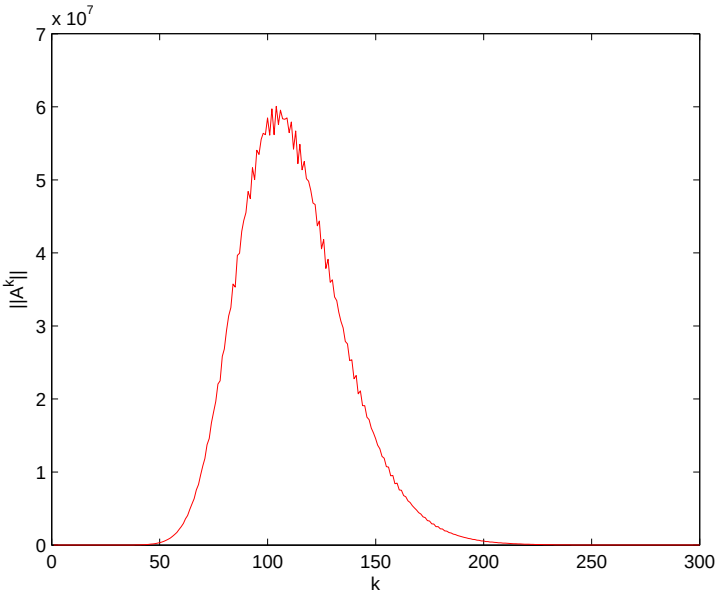

FIG. 2.2. *Behavior of* $\|A^k\|$ *for the matrix* A *(2.2)*

Notice that this is not a suitable algorithm for large matrices, due to expensive computation for each point of the grid.

3.2. Inverse power method. *An improvement of the Basic SVD algorithm can be done taking into account that just one singular value is needed and there is not need to compute all singular values of the matrix* $(zI - A)$*. For intance, applying the inverse power method to the matrix* $B = (zI - A)^H(zI - A)$ *we can obtain the smallest singular value of* $(zI - A)$*. This idea is presented in the next algotithm:*

```
for k=1:m
    for j=1:m
        B = (x(k)+y(j)*1i)*eye(N) - A);
        u = randn(N,1) + 1i*randn(N,1);
        [L,U] = lu(B);              %Compute LU factorization
        for p=1:maxit
            u = L'\U'\U\L\u; %Apply th inverse of B
            sig = 1/norm(u);
            if(abs(sigold/sig -1)< 1e-2) break; end
            u = sig*u;
            sigold = sig;
        end
    end
    sigmin(j,k) = sqrt(sig);
end
contour(x,y,log10(sigmin));
```

3.3. Lanczos method. *A more sophisticated method based in the inverse power method is the symmetric Lanczos method applied to the matrix* $(zI - A)^H(zI - A)$. *At every iteration, this method requires the solutions of linear systems with* $(zI - A)^H$ *and* $(zI - A)$, *for this reason one could previously perform a Schur factorization of the matrix A; this procedure will reduce the cost of each iteration, (see [12, 6]). This approach which is outlined in the next algorithm, can not be applied to large matrices due to its prohibitive cost.*

```
for k=1:m
    for j=1:m
        T1 = (x(k)+y(j)*1i)*eye(N) - A);
        T2 = T1';
        sigold = 0; qold = zeros(N,1); beta = 0; H = [];
        q = randn(N,1) + 1i*randn(N,1); q = q/norm(q);
        for p=1:maxit
            v = T1\(T2\q) - beta*qold;
            alpha = real(q'*v); v = v - alpha*q;
            beta = norm(v); qold = q; q = v/beta;
            H(p+1,p) = beta; H(p,p+1) = beta;  H(p,p) = alpha;
            sig = max(eig(H(1:p,1:p)));
            sigold = sig;
        end
        sigmin(j,k) = sqrt(sig);
    end
    sigmin(j,k) = sqrt(sig);
end
contour(x,y,log10(sigmin));
```

3.4. Projection Krylov subspaces methods. *In the large scale setting K. Toh and L. Trefethen proposed in [18], a projection onto a Krylov subspace of dimension* p *with* $p < n$ *:*

$$\mathcal{K}_p(A, x) = \{x, Ax, A^2x, \ldots, A^{p-1}x\},$$

using the Implicit restarted Arnoldi (IRAM)([16, 17]), to obtain factorizations of the form:

$$AV_p = V_{p+1}\bar{H}_p, \tag{3.1}$$

or

$$AV_p = V_pH_p, + f_m e_m^t \tag{3.2}$$

where V_p *is a* $n \times p$ *matrix with orthonormal columns which represent a basis of* $\mathcal{K}_p(A, x)$ *and* $\bar{H}_p$ *a upper Hessenberg matrix* $p+1 \times p$. *In their work Toh and Trefethen proved that:*

$$\Lambda_\epsilon(\bar{H}_p) \subseteq \Lambda_\epsilon(A).$$

In this case we are talking about the pseudospectrum of a rectangular matrix, where the definition 1.2 holds [25], then by calculating the pseudospectrum of the matrix $\bar{H}_p$ *one can approximate the set* $\Lambda_\epsilon(A)$.

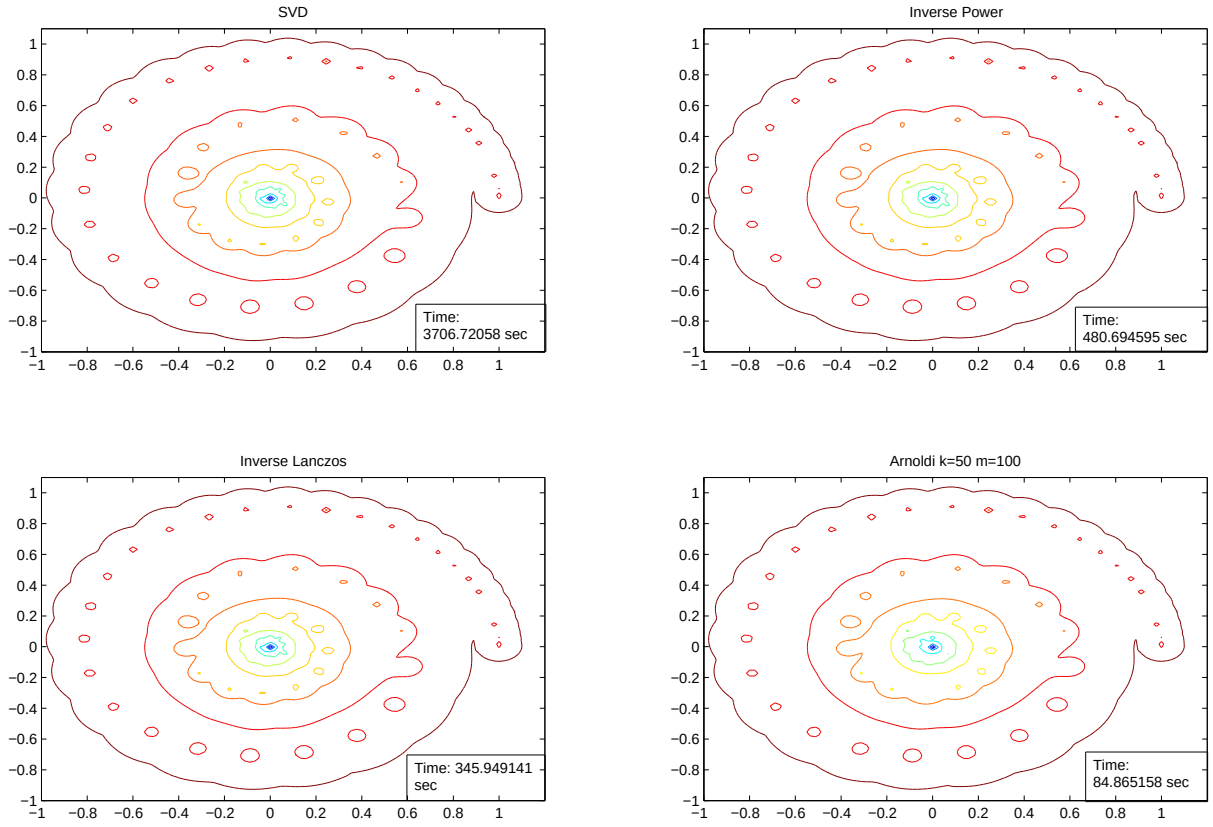

FIG. 3.1. *Comparison between classic methods to approximate the pseudospectrum of matrix of the discretization of 3.3*

In order to compare the methods previously mentioned, we consider the complex matrix associated with an integral equation arising in laser theory, which was investigated by Landau [11]:

$$Au(x) = \sqrt{iF/\pi} \int_{-1} 1e^{-iF(x-y)^2} u(y)dy \tag{3.3}$$

Figure (3.1) shows time (in seconds) required by Basic SVD, Inverse Power Method, Lanczos and Arnoldi iteration to compute the pseudospectra of the Landau matrix of 400×400*. As we can see, every method produce a similar approximation for the pseudospectrum, however, analyzing the time that each method require, we notice that, Arnoldi just took approximately 2% of the time used by the Basic SVD method, and about 24% of the time used by Inverse Lanczos. For a detailed discussion of the pseudospectra of this matrix see [20, 6]*

4. Block Arnoldi method. *This method builds a Block Krylov factorization of size m and block size b, such as:*

$$AV_{[m]} = V_{[m]}H_{[m]} + F_m E_m^t \tag{4.1}$$

or equivalently:

$$AV_{[m]} = V_{[m+1]}\bar{H}_{[m]} \tag{4.2}$$

where:

- $V_{[m]}^t AV_{[m]} = H_{[m]}$ *is a block upper Hessenberg matrix* $H_{[m]} \in \Re^{(m\times b)\times(m\times b)}$.
- $\bar{H}_{[m]} \in \Re^{((m+1)\times b)\times(m\times b)}$.
- $V_{[m]}^t V_{[m]} = I_{m\times b}$.
- $V_{[m]}^t F_m = 0$.
- *The matrix* $V_{[m]} = [V_1, V_2, \ldots, V_m]$, *contains blocks* $\{V_i\}_{i=1}^m$ *of size* $n \times b$, *which form an orthogonal basis of the block Krylov subspace of dimension* m*:*

$$\mathcal{K}_m(A, X) = \{X, AX, A^2X, \ldots, A^{m-1}X\},$$

  where X *is a* $n \times b$, *non-singular matrix.*

- $E_m = [\overbrace{Z_b, Z_b, Z_b, \ldots, Z_b}^{m-1}, I_b]$ *with* Z_b *and* I_b *the zero and identity matrices of order* b *respectively.*

The algorithm 1 shows the basic Block Arnoldi Method (see [15, 22] for more details).

Algorithm 1 Block Arnoldi method Modified Gram-Smith

```
1: Choose a unitary matrix V_1 of size n × b
2: for i = 1, ... do
3:     W_i = AV_i
4:     for j = 1, ... i do
5:         H_ji ← V_j^T W_i
6:         W_i ← W_i − V_j H_ji
7:     end for
8:     Compute the QR factorization of W_i
9:     V_{i+1} = Q,   H_{i+1,i} = R
10: end for
```

Different authors [5, 1, 14] have disccused advantages and disadvanteges for block methods, and these can be summarized as:

1. *The ability to compute multiple or clutered eigenvalues more efficiently.*
2. *High performance computing in large scale cases, block methods replace matrix-vector products with level-3 BLAS matrix-matrix multiplications.*
3. *Lost of orthonality beteween blocks.*
4. *For a fixed size of Krylov space. i.e. $m \times b$, a block method will attempt to find the wanted eigenspace in a matrix polynomial space of lower degree than the eigspace of single vector methods, which can lead to slower convergence for the block method.*
5. *Increased complexity of the algorithm and the lack of theoretical understanding.*

4.1. Block Implicit restarted Arnoldi. *In this work we are interested in analizing the performance of a block Arnoldi method in the computation of Pseudospectra for large matrices, and also its effectiveness compared with other methods. In the following sections a Block Implicitly Restarted Arnoldi (BLIRAM) will be described as well as its use in the computation of Pseudospectra for several test matrices.*

Suppose that we have m steps of a Block Arnoldi process, so we have $m+1$ $n \times b$-blocks $V_1, V_2, , V_m, F_m$, an upper Hessenberg Block Matrix $H_{[m]}$ and the equations 4.1 and 4.2 hold. It is prohibitive make $m \times b$ as large as n for large matrices, so is suitable to fix m and use the previus information to obtain a new starting block and restart the process to get a better approximation to the invariant subspace.
A generalization of the well-know Implicit Restarted Arnoldi to the block case was proposed by Z. Castillo in [5] (BLIRAM),

5. Projection Block Krylov methods for pseudospectra calculations. *As was described in [24] and [25], we use the block upper Hessenberg rectangular matrix $\bar{H}_{[m]}$ of dimension $(b \times m + 1) \times (b \times m)$, where b is the size block and m is the restarting parameter of the BLIRAM algorithm, to approximate the pseudosectrum of the matrix A, appliying the definition (1.2). Since the matrix $\bar{H}_{[m]}$ has been computed by BLIRAM with a suitable tolerance for the ritz values, for each point z of the region of interest, we calculate $s_{min}(zI - \bar{H}_{[m]})$ performing a QR factorization of $(zI - \bar{H}_{[m]})$ and the inverse Lanczos to the matrix $R^H R$ as we described in section 3.3. This procedure is outlined in algorithm 2.*

Algorithm 2 Bliram for pseudospectra calculations

1: Select Block size b, and restating parameter k, m
2: Call Bliram(k, m, b) to obtain the matrices $\bar{H}_{[m]}$ such as $AV_{[m]} = V_{[m+1]}\bar{H}_{[m]}$
3: Define a grid over a region of $\mathbb{C}$ enclosing converged Ritz values.
4: **for** each grid point z **do**
5: Compute the QR factorization of $zI - \bar{H}_{[m]}$
6: Get $\lambda_{max}(z)$ using inverse Lanczos on $R^H R$
7: $\sigma_{min}(z) = 1/\sqrt{\lambda_{max}(z)}$
8: **end for**

6. Numerical experiments. *In this secction we will compare the mainstream methods for pseudospectra calculation (Inverse Lanczos Iteration and Implicit restarted Arnoldi) with the block method proposed in the seccion 5. The parameters of Inverse Lanczos, IRAM and BLIRAM are decribed in the table 6.1.*

The experiment were performed in Matlab 7.3© on an Intel Pentium© IV 3.20GHz with 2GB of RAM running on Debian Linux. We compare the implicit restarted Arnoldi and block implicit restarted arnoldi, the stoping criteria for IRAM was based on the convergence of the ritz values produced for these both algorithms:

$$\|f\|_2\|D\|_2 < 1e-14,$$

where D is a diagonal matrix that contains the ritz eigenvalues. Stopping criterion in BLIRAM was:

$$\|F\|_F\|D\|_F < 1e-14,$$

TABLE 6.1
Parameters description: maxit is the maximun number of iteration allowed, initial guesses are chosen randomly.

Method	Parameters
Inverse Lanczos	maxit = 99
IRAM(k, m)	maxit = 10000 k = number of wanted eigenvalues m = size of the restart
BLIRAM(k, m, b)	maxit = 10000 b = block size k = number of wanted eigenvalues $k \times b$ m = size of the restart $m \times b$

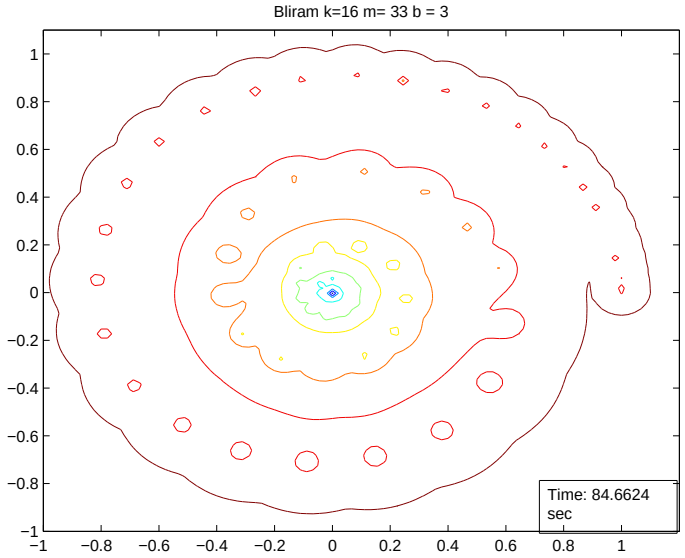

FIG. 6.1. *Pseudospectrum of matrix of the discretization of 3.3 calculated by BLIRAM*

where D is a diagonal matrix that contains the ritz values and $\| \star \|_F$ is the Frobenius norm. After these two process we approximate the pseudospectrum of the matrix A, with the pseudospectrums of the upper Hessenberg rectangular matrices obtained from IRAM and BLIRAM.

As IRAM, BLIRAM can approximate some portions of the pseudospectrum of a matrix, in this example we approximate the pseudospectrum of the discretization of the operator (3.3), the figure 6.1 is quite similar to the figures produced by Basic SVD, Inverse Power Method, Lanczos and Arnoldi iteration (figure 3.1), and this figure was produced by BLIRAM and IRAM in a similar time..

Projection methods have been widely used for approximate the pseudospectrum of large matrices [20, 24, 18]. In some cases they produce a good approximation in just a portion of the time required by inverse Lanczos method. For example, consider the

TABLE 6.2
Time comparison (bidiagonal matrix 6.1): Inverse Lanczos time: 63.3902 sec.

Method	Time (sec)	Method	Time (sec)
IRAM(2,8)	1.25969	BLIRAM(1,4,2)	1.41083
IRAM(4,10)	1.33441	BLIRAM(2,5,2)	1.52877
IRAM(6,12)	1.39961	BLIRAM(3,6,2)	1.38206
IRAM(8,14)	1.53168	BLIRAM(4,7,2)	1.36918

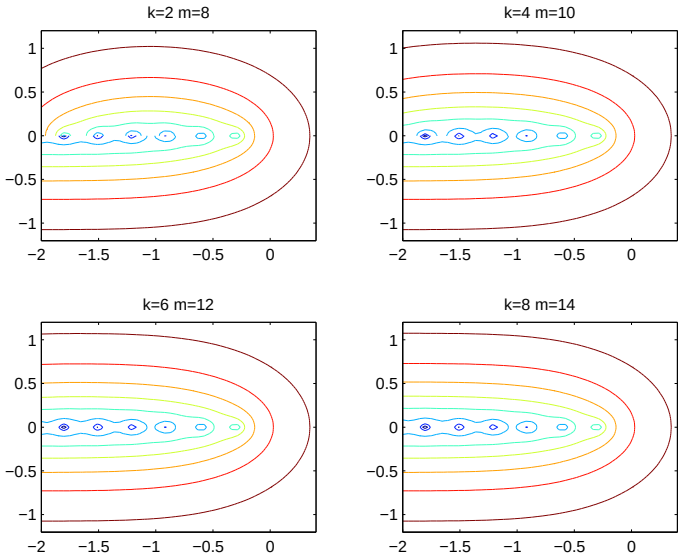

FIG. 6.2. *Pseudospectra of the bidiagonal matrix (6.1): IRAM (upper half) Inverse Lanczos (lower half)*

bidiagonal matrix defined in [18]:

$$a_{k,k} = -0.3k, \qquad a_{k+1,k} = 1 \tag{6.1}$$

Using this matrix of dimension 400 we evaluate the performance of the inverse Lanczos method, Iram and Bliram, in a grid of $[-2, 0.4] \times [-1.2, 1.2]$ *with* 50 *points equally spaced in each direction. As we can see, in the figures 6.2 and 6.3, the pseudospectrum generated by IRAM(8,14) and BLIRAM(4,7,2), fit with the pseudospectrum caculated by the inverse Lanczos iteration, also we can point out the improvement in the approximation of the pseudospectra while the restarting parameters increase. Computational time required by the projection methods is much lower than the time required by inverse Lanczos iteration (see table 6.2).*

In our next experiment we compute different levels of the pseudospectrum for matrix gre1107($n = 1107$*) from Matrix Market [3], and compare the performance of inverse lanczos iteration, IRAM(k,m), and BLIRAM(k,m,b), on the domain* $[-1, 1.5] \times [0, 1]$ *over a mesh of* 50×25 *points. In the figures 6.4, 6.5, 6.6, 6.7, 6.8 and , 6.9 we note a similar approximation of the pseudospectra generated by IRAM and BLIRAM,*

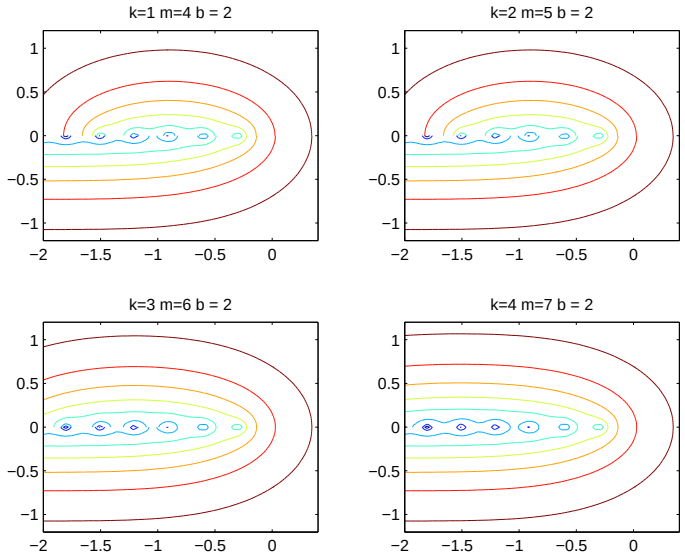

FIG. 6.3. *Pseudospectra of the bidiag matrix (6.1): BLIRAM (upper half) Inverse Lanczos (lower half)*

TABLE 6.3
Comparison in second for the differents method for the matrix gre1107

Method	Time (sec)
Inverse Lanczos	468.337
IRAM(100,150)	58.986
IRAM(120,240)	159.085
IRAM(150,270)	265.091
BLIRAM(33,50,3)	38.5291
BLIRAM(40,80,3)	107.799
BLIRAM(50,90,3)	149.108

just a crude approximation to the pseudospectra generated by Inverse Lanczos algorithm for $\epsilon = 10^{-2}$*, and a better a approximation for* $\epsilon = 10^{-1}$*, but BLIRAM took a lower time (see table 6.3).*

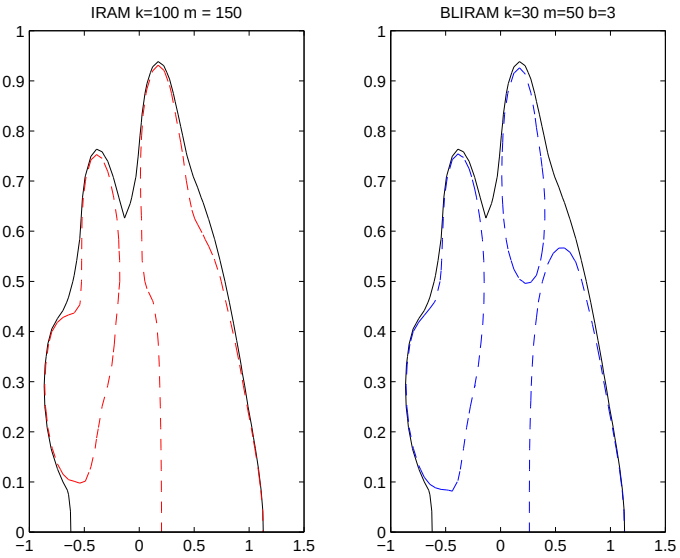

FIG. 6.4. *Pseudospectra of gre1107 matrix* $\epsilon = 10^{-1}$

In order to ilustrate the strongly dependency of the location of the Ritz values as BLIRAM aproximates the pseudospectrum of a matrix, we use the pentadiagonal Toeplitz matrix:

```
A = gallery('toeppen',200,0,1/2,0,0,1).
```

The pseudspectrum of this matrix is calculated over the region $[-2.5, 2.5] \times [-2.5, 2.5]$ *over a mesh of* 50×50 *points. As is depicted in figures 6.10 and 6.11, BLIRAM has a similar behavior that IRAM, the approximations of the pseudospectra is more accurate near the converged ritz value, i.e, the approximation of pseudospectrum is just locally acceptable for both method. A possible strategy to approximate the complete pseudospectra could be the repeated application of BLIRAM and IRAM, with different eigenvalues selction criteria.*

One of the qualities of the matrix A which affects the pseudospectra obtained by the Krylov projection methods, is the non-normality of the matrix A [24], an example of a slightly nonnormal matrix is the Jacobi matrix of dimension 800 for the reaction-diffusion Brusselator model (rb800l) from chemical engineering [8]. Obeserving the figure 6.12 when one seeks the rightmost eigenvalues with IRAM and BLIRAM the approximations of the pseudospectrum are very close to the pseudospectrum produced by inverse Lanczos iteration (grid on $[-1.2, 1.2] \times [-1, 2.5]$ *with* 50×60 *points). On the other hand, experimenting with the highly non-normal matrix GRCAR [24] of order 800 , both projection methods do not produce a good approximation of the pseudospectra (see figure 6.13).*

7. Conclusions. *In this work we studied the performance of a Block Implicit restarted Arnoldi method for pseudospectra calculations of large matrices. Numerical experimentation shows:*

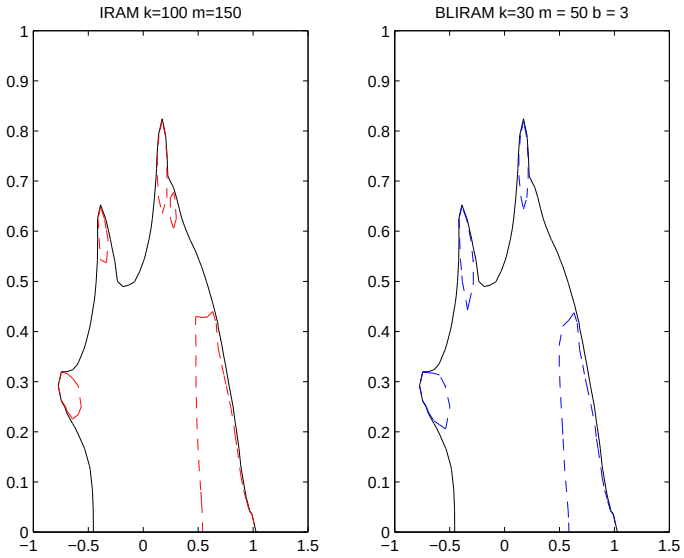

FIG. 6.5. *Pseudospectra of gre1107 matrix* $\epsilon = 10^{-2}$

TABLE 6.4
Comparison in second for the differents method for the matrix rb800l

Method	Time (sec)
Inverse Lanczos	603.134
IRAM(90,150)	148.623
BLIRAM(30,50,3)	84.5917

- *The block Arnoldi method studied in this work (BLIRAM) has a similar behavior to the classical projection method used for pseudospectra calculation, the Implicit restarted Arnoldi (IRAM).*
- *The quality of the pseudospectra plotted using IRAM and BLIRAM are quite similar.*
- *BLIRAM generally approximates the pseudspectrum of our test matrices in a lower time than IRAM.*

However there are several points that require attention:

- *BLIRAM as all projection methods just approximate the pseudospectrum, and it has a strongly dependency of the location of the ritz values.*
- *Block Krylov methods can fail when approximate the pseudospectrum of highly non-normal matrices.*

Finally, the next step we want to consider in the near future is to combine BLIRAM with various acceleration and preconditioned methods and to use this Block Arnoldi method for computation of the pseudospectra of matrices arising in some interesting applications in sciences a engineering.

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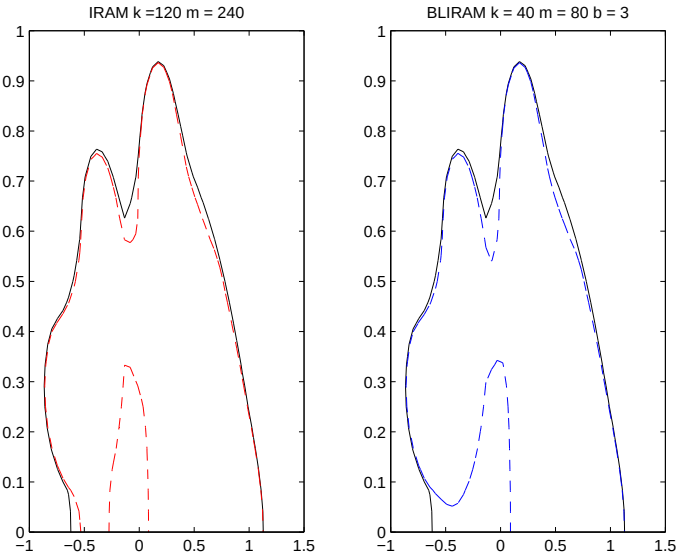

FIG. 6.6. *Pseudospectra of gre1107 matrix* $\epsilon = 10^{-2}$

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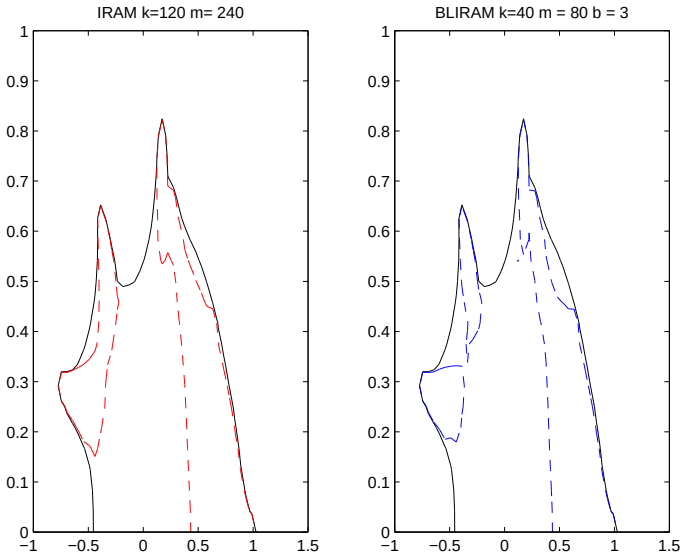

FIG. 6.7. *Pseudospectra of gre1107 matrix* $\epsilon = 10^{-2}$

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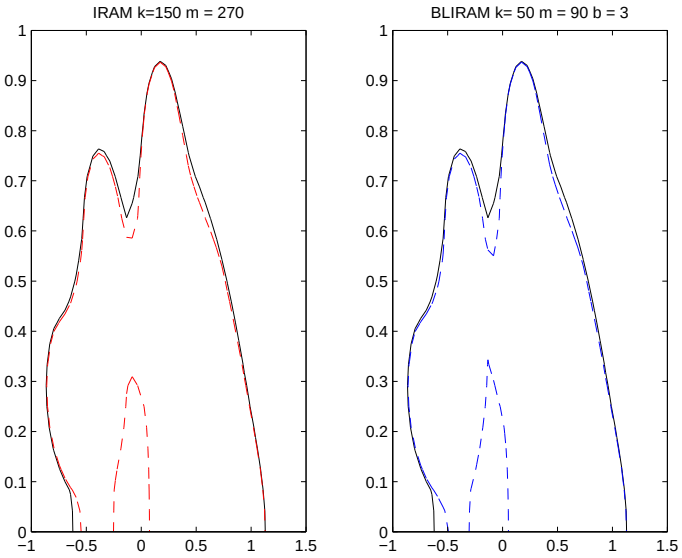

FIG. 6.8. *Pseudospectra of gre1107 matrix* $\epsilon = 10^{-2}$

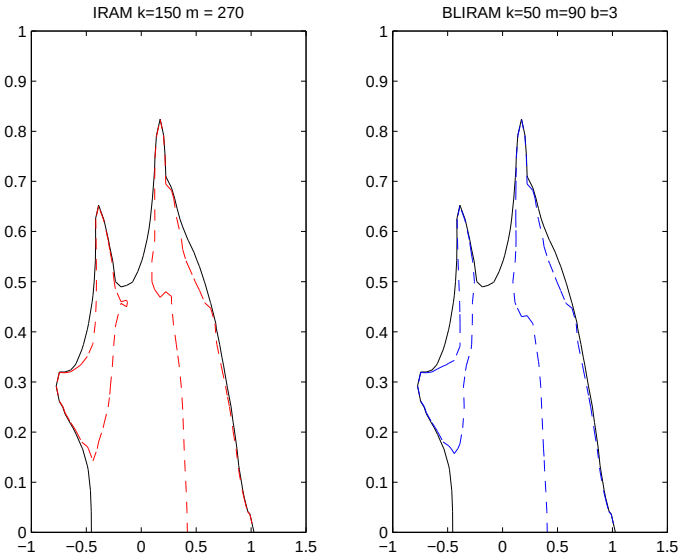

FIG. 6.9. *Pseudospectra of gre1107 matrix* $\epsilon = 10^{-2}$

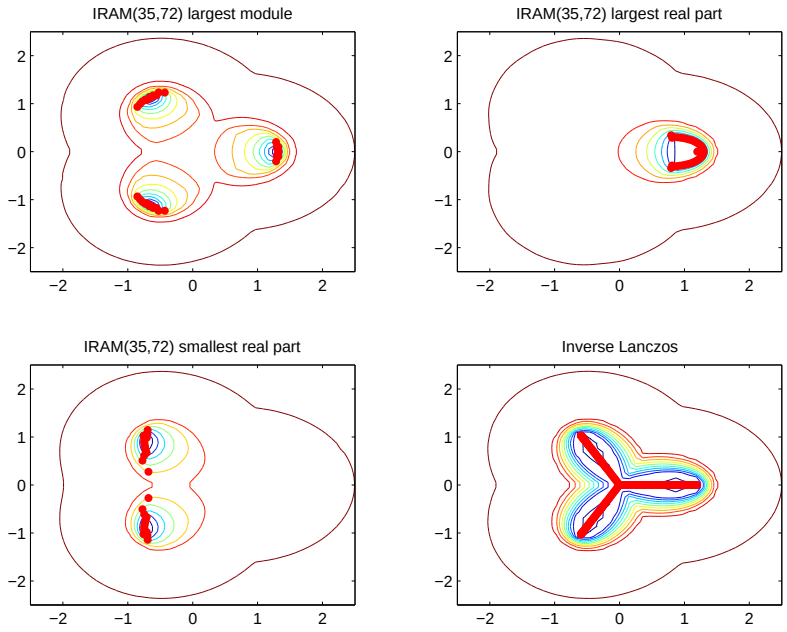

FIG. 6.10. *Effect over the pseudospectrum calculated by IRAM of pentadiagonal Toeplitz matrix and the localization of ritz values*

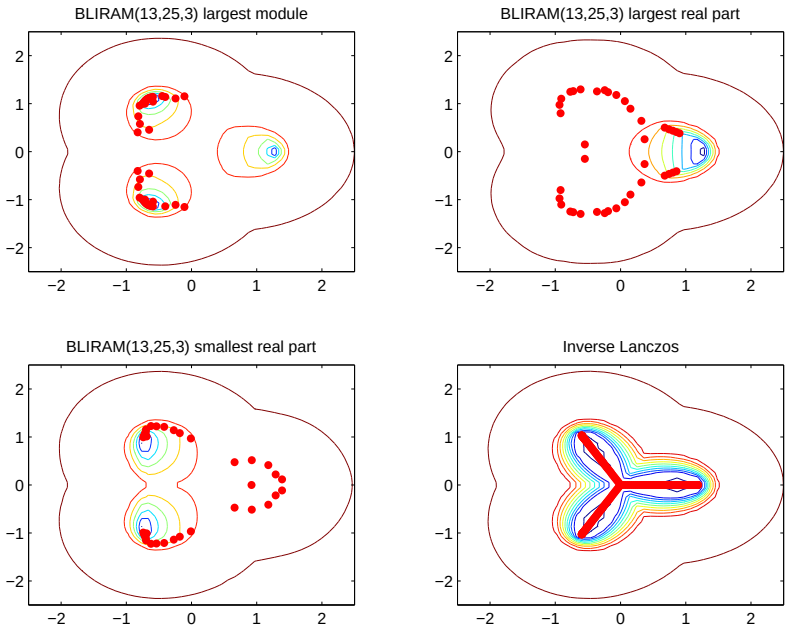

FIG. 6.11. *Effect over the pseudospectrum calculated by BLIRAM of pentadiagonal Toeplitz matrix and the localization of ritz values*

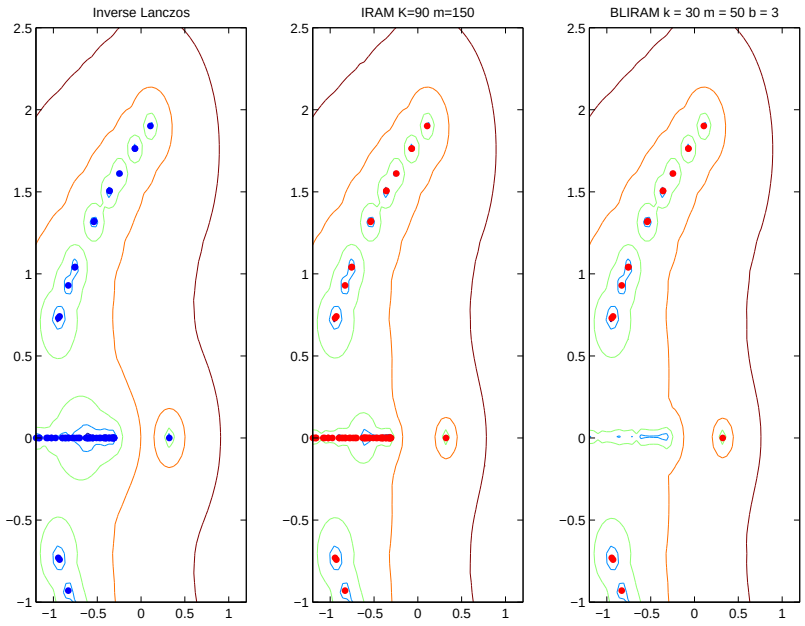

FIG. 6.12. *Pseudospectra of the matrix rb800l: Inverse Lanczos (left), IRAM (Center), BLIRAM (right)*

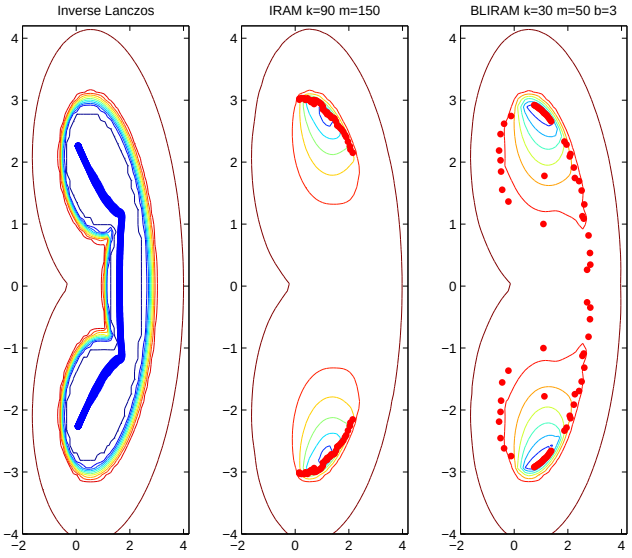

FIG. 6.13. *Pseudospectra of the matrix GRCAR800: Inverse Lanczos (left), IRAM (Center), BLIRAM (right)*