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A domain decomposition scheme based on a discretisation without crosspoints and alternating oblique projections

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Abstract

In this work, we propose a domain decomposition resolution scheme for partial differential equations. In this scheme, the discrete Lagrange multiplier is forced to vanish into cross-points between artificial interfaces. The resulting saddle point problem is then solved efficiently by a recently developed alternating oblique projection method. The properties of the proposed discretisation scheme will be illustrated solving Poisson equations.

AMS: 65F10, 65N22, 65Y05.
Keywords: Domain decomposition, alternating projections, saddle point problems, Poisson equation.

1 Introduction

The numerical solution of partial differential equations (PDE) often leads to large-scale algebraic systems. Parallel computer architectures are frequently used for solving these large-scale systems, but for taking advantage of parallel machines it is often necessary to reorganize the problem in a suitable way. Domain decomposition is a problem formulation strategy that takes advantage of the presence of several processors in a natural way. Any domain decomposition method is based on the assumption that the given computational domain, say Ω, is partitioned into subdomains $\Omega_i, i = 1, \ldots, k$. Then, the original problem can be reformulated upon each subdomain Ω_i, yielding a family of subproblems of reduced size that are coupled one to another through the values of the unknown solution at the subdomain interfaces. [KST95] [QV99], [GPP95].

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Discrete formulation by the domain decomposition of a linear PDE usually leads to a saddle point linear system, of the following form:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{1}$$

In [HR05] an iterative method is proposed for solving saddle point problems (1) in a computational efficient way, under the following assumptions:

- the computational cost of solving $Ay = b$ is reasonable and,
- the orthogonal projection P onto the null space of B ($\ker B$) is easily obtained.

In a typical domain decomposition strategy, the matrix A is a block-diagonal matrix where each block diagonal $A_i, i = 1 \ldots, k$, represents a local operator in the subdomain Ω_i. The B matrix represents the coupling conditions between interfaces. (v.f [QV99]). When parallel machines are used, each block A_i can be assigned to each processor finding the solution of linear systems $Ax = y$ very efficiently. However, due to the cross-points between artificial interfaces of sub-domains, coupling conditions between interfaces can not be represented by a block-diagonal matrix B. Consequently, low-cost orthogonal projections onto $\ker B$ can not be obtained. In this work, we propose a domain decomposition discrete formulation that forces the Lagrange multiplier of the coupling conditions to vanish at cross-points. With a convenient numerical integration formula, the matrix B that represents the coupling conditions becomes a block-diagonal matrix and the orthogonal projection onto $\ker B$ is easily obtained. Finally, the AOP method [HR05] is used for solving the resulting saddle point problem efficiently.

2 The model problem

Let Ω be an open bounded subset of $\Re^2$, with a rectangular external boundary Γ_0. We propose to solve in Ω the following Poisson type problem with Dirichlet boundary conditions,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where f is a given function in Ω.
It is well-known that if f is sufficiently regular then problem (2) has a unique solution (for details see e.g., [JOH92]).

3 Domain decomposition formulation

The domain decomposition formulation is given by the partition of Ω in four sub-domaines $\Omega_i,\ i = 1, \ldots, 4$ as showed in figure 1. With this notation, the model problem (2) reduces to four coupled problems (with indices numbered

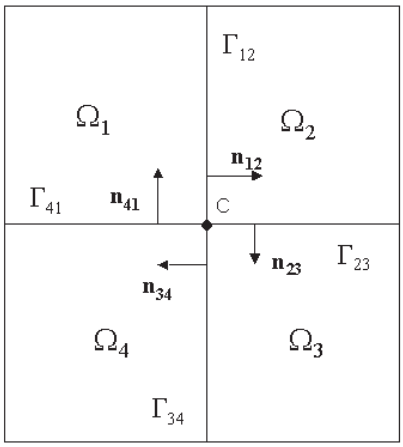

Figure 1: Decomposition of Ω

modulo 4).

Find $u_i,\ \lambda_{i,i+1},\ i = 1, 2, 3, 4.$

$$
\begin{aligned}
-\Delta u_i &= f \text{ in } \Omega_i && (3)\\
u_i &= 0 \text{ on } \Gamma_i = \partial\Omega \cap \partial\Omega_i && (4)\\
u_i &= u_{i+1} \text{ on } \Gamma_{i,i+1} && (5)\\
\frac{\partial u_i}{\partial n_{i,i+1}} &= \lambda_{i,i+1} \text{ on } \Gamma_{i,i+1} && (6)\\
\frac{\partial u_i}{\partial n_{i-1,i}} &= \lambda_{i-1,i} \text{ on } \Gamma_{i-1,i} && (7)
\end{aligned}
$$

Model problem (2) and problem (3-7) are equivalent. This means that $u_i = u\,|_{\Omega_i},\ i = 1, 2, 3, 4$ where u is the solution of model problem (2).
Remark: The unknowns $\lambda_{i,i+1},\ i = 1 \ldots, 4$ are called the Lagrange's multipliers associated with the artificial boundary condition $u_i = u_{i+1}$ on $\Gamma_{i,i+1}$ in the corresponding saddle point approach.

4 Domain decomposition variational formulation

We assume that $f \in L^2(\Omega)$. Next, we will define a variational formulation for problem (3-7). For this, we also assume that the solution u of (2) is sufficiently regular to guarantee the existence of the normal derivative of u at each $\Gamma_{i,i+1}$. We assume $u \in H^{1+\epsilon}(\Omega)$ with $\epsilon \in (0, \frac{1}{2}]$, therefore for $i = 1, 2, 3, 4$ the traces of u_i belong to $H^{1/2+\epsilon}(\partial\Omega_i)$, and the normal derivatives $\frac{\partial u_i}{\partial n_i} \in H^{-1/2+\epsilon}(\partial\Omega_i)$.

Hence, $\frac{\partial u_i}{\partial n_i}$ on $\Gamma_{i,i+1}, \Gamma_{i-1,i}$ belongs to $H^{-1/2}(\Gamma_{i,i+1}, \Gamma_{i-1,i})$ and the components u_i of u satisfy equations (3-7).

If we take the scalar product of each equation (3-4) with the convenient test functions v_i, defined in Ω_i (whose trace vanishes in $\partial\Omega_i \cap \partial\Omega$), and applying Green's formula, we obtain, for $i = 1$,

$$\int_{\Omega_1} f v_1 dx = \int_{\Omega_1} \nabla u_1 \nabla v_1 dx - \int_{\Gamma_{12}} \frac{\partial u_1}{\partial n_{12}} v_1 d\sigma + \int_{\Gamma_{41}} \frac{\partial u_1}{\partial n_{14}} v_1 d\sigma. \tag{8}$$

In general for $i = 1, 2, 3, 4$, it holds

$$\int_{\Omega_i} f v_1 dx = \int_{\Omega_i} \nabla u_i \nabla v_i dx - \int_{\Gamma_{i,i+1}} \frac{\partial u_i}{\partial n_{i,i+1}} v_i d\sigma + \int_{\Gamma_{i-1,i}} \frac{\partial u_i}{\partial n_{i-1,i}} v_i d\sigma, \tag{9}$$

Adding these equations and using (6) and (7) it follows that

$$\sum_{i=1}^{4} \{ \int_{\Omega_i} \nabla u_i \nabla v_i dx + \int_{\Gamma_{i,i+1}} \lambda_{i,i+1}(v_{i+1} - v_i) d\sigma \} = \sum_{i=1}^{4} \int_{\Omega_i} f v_i dx. \tag{10}$$

This corresponds to the first equation of the variational formulation. To obtain the second equation, the ones that force the continuity of u_i, we take the scalar product of (5) with the test functions $\mu_{i,i+1}$ and, then, we summarize them for $i = 1, 2, 3, 4$, and obtain

$$\sum_{i=1}^{4} \int_{\Gamma_{i,i+1}} \mu_{i,i+1}(u_{i+1} - u_i) d\sigma = 0. \tag{11}$$

Now, we are ready to write the variational formulation of the problem. We take for $i = 1, 2, 3, 4$

$$V_i = \{ v_i \in H^1(\Omega_i); v_i \mid \Gamma_i = 0 \} \text{ and } V = \Pi_{i=1}^{4} V_i, \tag{12}$$

where $\Gamma_i = \partial\Omega_i \cap \partial\Omega$, equipped with the norm defined by

$$\|v\|_V = (\sum_{i=1}^{4} \|v_i\|^2_{H_1(\Omega_i)})^{1/2} \tag{13}$$

and

$$M = \Pi_{i=1}^{4} H^{-1/2}(\Gamma_{i,i+1}) \tag{14}$$

equipped with the norm defined by

$$\|\mu\|_M = (\sum_{i=1}^{4} \|\mu_{i,i+1}\|^2_{H^{-1/2}(\Gamma_{i,i+1})})^{1/2}. \tag{15}$$

On $V \times V$, we define the following bi-linear forms,

$$\forall v \in V, \forall w \in V : a(v, w) = \sum_{i=1}^{4} \int_{\Omega_i} \nabla v_i \nabla w_i dx, \tag{16}$$

$$\forall v \in V :< f, v >= \sum_{i=1}^{4} \int_{\Omega_i} f v_i dx, \tag{17}$$

and on $M \times V$ we define the bi-linear form,

$$\forall \mu \in M, \forall v \in V : b(\mu, v) = \sum_{i=1}^{4} \int_{\Gamma_{i,i+1}} \mu_{i,i+1}(v_i - v_{i+1}) d\sigma. \tag{18}$$

Hence, we obtain the following mixed problem,
Find $(u, \lambda) \in V \times M$ that

$$\forall v \in V : \qquad a(u, v) - b(\lambda, v) =< f, v > \tag{19}$$

$$\forall \mu \in M : \qquad b(\mu, u) = 0. \tag{20}$$

Summing up, we have established the following result.

Proposition 4.1 *The solution u of model problem(2) is a solution of (3-7) and this solution verifies the variational formulation (19-20).*

5 Proposed Discretisation

For the discretization, we build an independent regular triangulation T_i^h in each sub-domain Ω_i for $1 \leq i \leq 4$, so that each T_i^h is conforming, but globally the union of triangulations is not conforming because they do not necessarily match at the intersections of the interfaces. We denote by h_i the discretisation parameter in Ω_i. We now introduce the finite dimensional spaces,

$$V_i^h = \{v_i^h \in C^0(\bar{\Omega}_i) \text{ that } \forall T \in T_i^h, v_i^h|_T \in \mathbb{P}_1, v_i^h|_{\Gamma_i} = 0\} \tag{21}$$

$$V_h = \prod V_i^h, \tag{22}$$

where $\mathbb{P}_1$ is the two variables polynomial space of degree ≤ 1. We note here that $V_i^h \subset V_i$ and $V_h \subset V$. We also introduce the discrete space for approaching the Lagrange's multiplier space. For that we denote by C the cross-point of interfaces $\Gamma_{i,i+1}$ as indicated in Figure 1. In each interface $\Gamma^{i,i+1}$, one chooses the mesh for the Lagrange multiplier as the restriction of $T^h|_{\Gamma_{i,i+1}}$ with fewest number of nodes on the interface. The mesh on interface $\Gamma_{i,i+1}$ is denoted by $S_\eta^{i,i+1}$, and $\eta_{i,i+1}$ is the discretisation parameter. Finally, we introduce the spaces:

$$W_\eta^{i,i+1} = \{\lambda_\eta \in C^0(\bar{\Gamma_{i,i+1}}); \forall S \in S_\eta^{i,i+1}, \lambda_\eta|_S \in \mathbb{P}_1, \tag{23}$$
$$\lambda_\eta \text{ vanish at the ends of } \Gamma_{i,i+1}\}$$

$$W = \prod_{i=1}^{4} W_\eta^{i,i+1}. \tag{24}$$

We would like to stress out that we are forcing the Lagrange multiplier approximation denoted by $\lambda_\eta^{i,i+1}$ to vanish into cross-points C and in points of

$\partial\Omega \cap \Gamma_{i,i+1}$. The dimension of the space $W_\eta^{i,i+1}$ is denoted by $M_{i,i+1}$. We denote by $\{x_k^{i,i+1}\}_{k=1}^{M_{i,i+1}}$ the set of nodes of the mesh $S_\eta^{i,i+1}$ and $v(x_k)$ and $\lambda(x_k)$ the values of v and λ at these points. For this discretization, on each interface $\Gamma_{i,i+1}, i = 1, 2, 3, 4$, we use the composed trapezoidal quadrature:

$$S_{i,i+1}(\lambda, v) = \sum_{k=0}^{M_{i,i+1}-1} \frac{\eta_{S_k}}{2}[v(x_k^{i,i+1})\lambda(x_k^{i,i+1}) + v(x_{k+1}^{i,i+1})\lambda(x_{k+1}^{i,i+1})].$$

We define

$$b_h(\lambda, v) = \sum_{i=1}^{4} S_{i,i+1}(\lambda, v^{i+1} - v^i),$$

and now we can write the discrete problem:
Find $(u_h, \lambda_\eta) \in V_h \times W_\eta$ that:

$$\forall v_h \in V_h : a(u_h, v_h) - b_h(\lambda_\eta, v_h) =< f, v_h > \tag{25}$$

$$\forall \mu_\eta \in W_\eta : b_h(\mu_\eta, u_h) = 0. \tag{26}$$

We can write (25) as a saddle point linear system [BRE74]:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{27}$$

where matrix

$$A = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix}, \tag{28}$$

with a re-numbering of nodes indices, we can write the matrix B as:

$$B = \begin{pmatrix} B_{12} & 0 & 0 & 0 \\ 0 & B_{23} & 0 & 0 \\ 0 & 0 & B_{34} & 0 \\ 0 & 0 & 0 & B_{41} \end{pmatrix}, \tag{29}$$

and right hand side vector

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}. \tag{30}$$

We can observe that A and B are block-diagonal matrices. The blocks $B_{i,i+1}$ are related to the discretisation of the coupling conditions at each interface $\Gamma_{i,i+1}$. Since all matrices $B_{i,i+1}$ are full-rank matrices, then the matrix B is also a full rank matrix. The matrix A_i corresponds to the discretisation of the local Laplacian operator at each subdomain Ω_i. Hence, the matrix A is symmetric and positive definite. Therefore, we obtain the following result.

Proposition 5.1 *The system (25-26) has a unique solution.*

6 The alternating oblique projections method

The alternating oblique projection (GC-AOP) method has been recently proposed in [HR05] for solving saddle point linear systems (1), and fits nicely with the discretisation described above. Indeed, since B is now a block diagonal matrix then computing the projection onto $\ker B$ can be done independently for each block $B_{i,i+1}$, which significantly reduces the required computational work for building the preconditioner. The GC-AOP method is equivalent to solving the condensed system $BA^{-1}B^T\lambda = BA^{-1}F$ combined with a version of the preconditioned conjugate gradient method. The preconditioner $Q_B^{-1} = (B^+)^T AB^+$ for the Schur complement matrix $BA^{-1}B^T$ is obtained alternating oblique projections onto $\ker B$ with projections onto the linear variety defined by:

$$V = \{x/f - Ax \perp \ker B\}. \tag{31}$$

We write here the alternating oblique projection algorithm for the coupled linear system:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{32}$$

In here, $R = I - P$ represents the projection matrix onto the orthogonal complement of $\ker B$.

1. **initialization:**
   - $x_0 \in V$
   - $w_0 = RARx_0$
   - $y_0 = w_0$
2. **iteration: For** $k = 0, 1, \ldots,$ **Do**
   - $d_k = A^{-1}y_k$
   - $\alpha_k = \frac{\langle x_k, w_k \rangle}{\langle d_k, y_k \rangle}$
   - $x_{k+1} = x_k - \alpha_k d_k$
   - $w_{k+1} = RARx_{k+1}$
   - $\beta_{k+1} = \frac{\langle x_{k+1}, w_{k+1} \rangle}{\langle x_k, w_k \rangle}$
   - $y_{k+1} = w_{k+1} + \beta_{k+1} y_k$
3. **End**

7 Numerical Experiments

In this section we describe the results obtained when solving a discrete nonconformal grid system (25) by the GC-AOP method. The vector f corresponds to the exact solution of equation (2) which we have set as:

$$u(x, y) = sin(K_1 \pi x) sin(K_2 \pi y) + 3,$$

with $K_1 = 1.7, K_2 = 2.3$, in $\Omega = [-1,1] \times [-1,1]$. All our experiments were run on a Pentium IV at 3.4 Mhz using MATLAB 7.0.

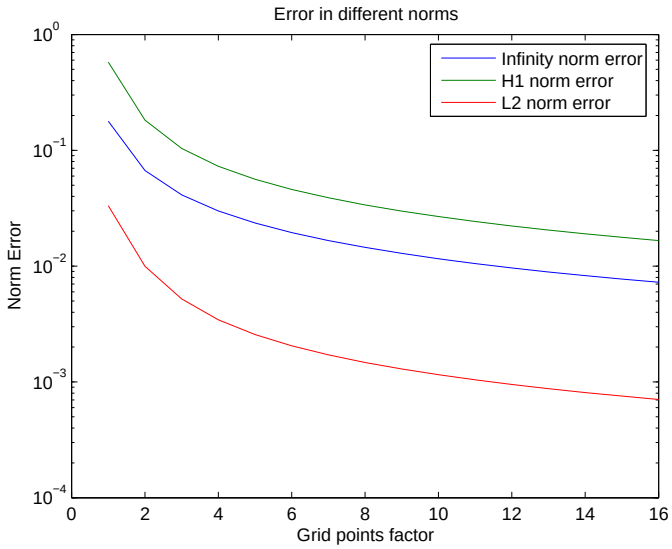

Figure 2: Norm of the error for a grid of $(k*15) \times (k*10)$ nodes in Ω_1, $(k*10) \times (k*15)$ nodes in Ω_2, $(k*18) \times (k*12)$ nodes in Ω_3 and $(k*15) \times (k*20)$ nodes in Ω_4. In here k represents the Grid point factor.

Figure 2 shows the evolution (measured in different norms), of the relative global error as a function of the mesh size. The comparison has been performed for a grid of $(k*15) \times (k*10)$ nodes in Ω_1, $(k*10) \times (k*15)$ nodes in Ω_2, $(k*18) \times (k*12)$ nodes in Ω_3 and $(k*15) \times (k*20)$ nodes in Ω_4. When the discretisation parameters tend to zero (i.e., when k increases), we can observe convergence of the discrete model to the exact solution of problem (2).

Figure 3 shows the relative error for a non-conformal grid and a mesh with 75×50 nodes in Ω_1, 50×75 nodes in Ω_2, 90×60 nodes in Ω_3 and 75×100 nodes in Ω_4. From Figure 3 we notice that the relative error is higher at the crosspoints between sub-domains and on the interfaces. However, as we can see from Figure 2, this fact does not affect the convergence of the method.

Figure 4 compares preconditioning time, with assembling time and the time required to perform 10 iterations of the CG-AOP method. We observe that when the discretisation parameters decrease (i.e., when k increases), the preconditioning time becomes insignificant with respect to the iteration time and the assembling time. This fact is due to the inexpensive cost of computing the projections onto $\ker B$ for building the preconditioner.

In Figure 5 we show the evolution of the error during the convergence process of the CG-AOP method, for different grid sizes. We can observe the convenient behavior of the method when the grid size increases.

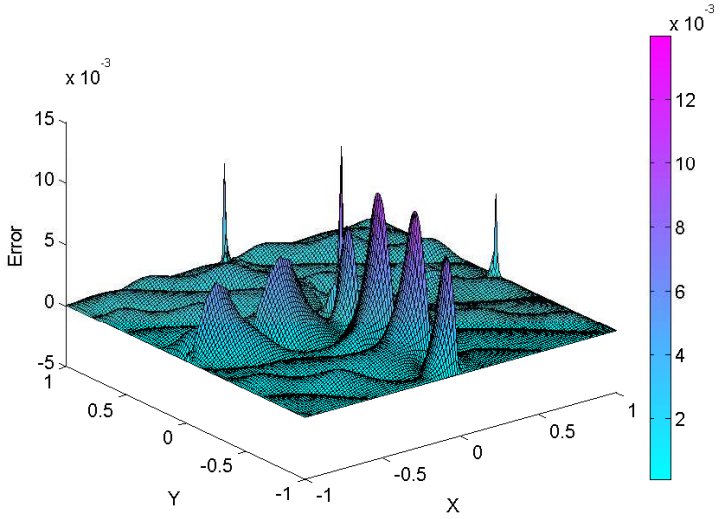

Figure 3: Relative error for a non-conformal grid and a mesh with 75×50 nodes in Ω_1, 50×75 nodes in Ω_2, 90×60 nodes in Ω_3 and 75×100 nodes in Ω_4.

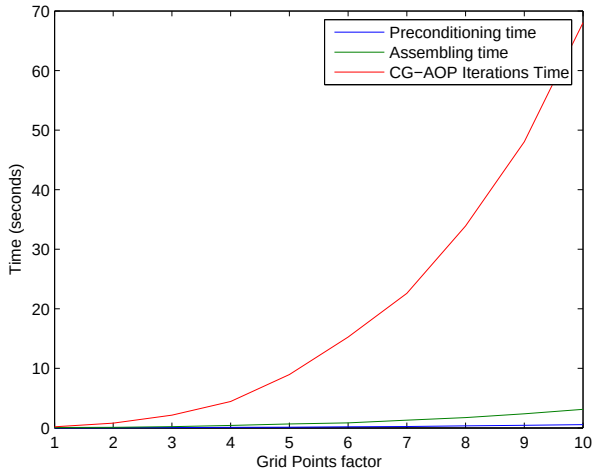

Figure 4: Assembling time, preconditioning time, and time required for 10 iteration on a grid of $(k*15) \times (k*10)$ nodes in Ω_1, $(k*10) \times (k*15)$ nodes in Ω_2, $(k*18) \times (k*12)$ nodes in Ω_3 and $(k*15) \times (k*20)$ nodes in Ω_4, where k is the grid points factor.

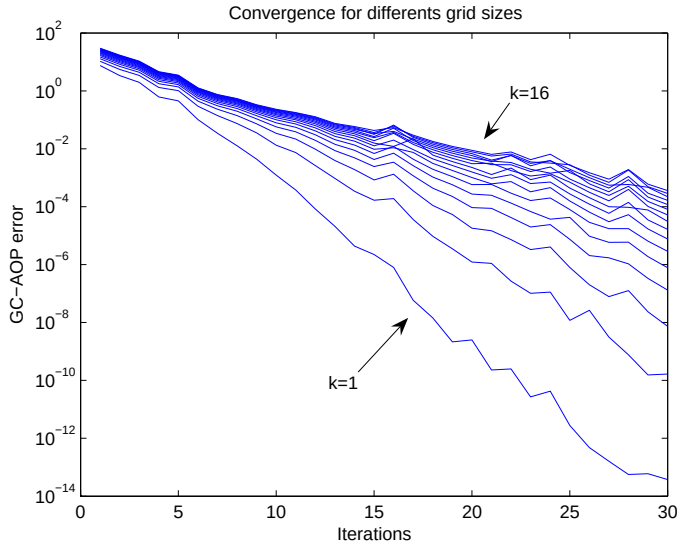

Figure 5: Convergence of CG-AOP for a grid of $(k*15)\times(k*10)$ nodes in Ω_1, $(k*10)\times(k*15)$ nodes in Ω_2, $(k*18)\times(k*12)$ nodes in Ω_3 and $(k*15)\times(k*20)$ nodes in Ω_4, where k is the grid points factor.

8 Conclusions

The proposed discretisation combined with the CG-AOP method produces a very efficient technique for solving Poisson type equations. The preconditioner can be built inexpensively and independently for each interface between sub-domains.

In case of using parallel machines, each interface block matrix B_{ij} could be assigned conveniently between processors for minimizing the communication cost. For this proposed discretisation, not only the work associated with the sub-domains can be distributed between processors, but also the work related to the interfaces can be parallelized.

As a continuation of this research, we would like to analyze the performance of the proposed scheme on suitable parallel machines, and also to establish the theoretical order of convergence of the proposed discretisation.

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