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Abstract

The Matrix Model Updating Problem (MMUP), considered in this paper, concerns updating a symmetric second-order finite element model so that the updated model reproduces a given set of desired eigenvalues and eigenvectors by replacing the corresponding ones from the original model, and preserves the symmetry of the original model. In an optimization setting, this is a constrained nonlinear optimization problem. Taking advantage of the special structure of the constraint sets, it is first shown that the MMUP can be formulated as an optimization problem over the intersection of some special subspaces and linear varieties on the space of matrices. Using this formulation, an alternating projection method is then proposed and analyzed. The projections onto the involved subspaces and linear varieties are characterized. To the best of our knowledge, an alternating projection method for MMUP has not been proposed in the literature earlier. A distinct practical feature of the proposed method is that it is implementable using only a few measured eigenvalues and eigenvectors. No knowledge of the eigenvalues and eigenvectors of the associated quadratic matrix pencil is required. The results of our numerical experiments on both illustrative and benchmark problems show that the algorithm works well. The paper concludes with some future research problems.

1 Introduction

It is well-known that vibrating structures, such as bridges, highways, buildings, etc., can be mathematically modelled by a system of differential equations of the form:

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = 0, \tag{1}$$

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where M, D and K are $n \times n$ matrices; and $\dot{x}(t)$ and $\ddot{x}(t)$ denote the first and second derivatives of the time-dependent vector $x(t)$, respectively.

Equation (1) is usually obtained by discretization of a distributed parameter system with finite element techniques, and therefore, known as the finite element model. The matrices M, D, and K are known as mass, damping and stiffness matrices, respectively. They are often very large and sparse, but have some nice structures, such as, M is symmetric and positive definite ($M = M^t > 0$), D and K are symmetric ($D = D^t, K = K^t$). Through out the whole paper, such assumptions will be made.

Assuming that the solutions $x(t)$ of (1) are of the form $x(t) = ve^{\lambda t}$, the scalar λ and the vector v solve the quadratic eigenvalue problem (QEP):

$$(\lambda^2 M + \lambda D + K)v = 0,$$

which has $2n$ eigenvalues and $2n$ eigenvectors. The eigenvalues are the roots of the nonlinear equation $det(P(\lambda)) = 0$ where

$$P(\lambda) = \lambda^2 M + \lambda D + K. \tag{2}$$

The eigenvalues of $P(\lambda)$ are related to the natural frequencies of the homogeneous system and the eigenvectors are the mode shapes of the vibration of the system (see, e. g., [11],[12], [30]). The dynamics of the system are modelled by these eigenvalues and eigenvectors. For example, sometimes the vibrating structures experience dangerous vibrations, called *resonance*, when a natural frequency becomes close or equal to a frequency of an external force, such as earthquake, gusty wind, weights of the human bodies, etc. Similarly, the stability of a vibrating system is determined by nature of a few dominating natural frequencies. The solutions of these problems naturally lead to the following inverse eigenvalue problem for the quadratic matrix pencil: Given

- Real $n \times n$ matrices M, K, D ($M = M^t > 0$, $K = K^t$ and $D = D^t$), where the spectrum of (2) is $\{\lambda_1, \ldots, \lambda_{2n}\}$ and eigenvectors are $\{x_1, \ldots, x_{2n}\}$.
- A set of p self-conjugate numbers, $\{\mu_1, \ldots, \mu_p\}$ and p vectors $\{y_1, \ldots, y_p\}$, where $p < 2n$,

find matrices $\tilde{K}, \tilde{D} \in \mathbb{R}^{n\times n}$ ($\tilde{K} = \tilde{K}^t$ and $\tilde{D} = \tilde{D}^t$), such that the spectrum of $\tilde{P}(\lambda) = \lambda^2 M + \lambda\tilde{D} + \tilde{K}$ is $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and eigenvectors are $\{y_1, \ldots, y_p, x_{p+1}, \ldots, x_{2n}\}$. The last requirement, namely, the invariance of the last $2n - p$ eigenvalues and the corresponding eigenvectors, is known as the *no spill-over* in vibration literature. The problem is called *Quadratic Eigenvalue Assignment Problem*(QPEVAP), when it is solved using feedback control techniques. A usual approach for solving the QPEVAP is to transform the problem to a standard first-order state space problem and then apply some of the specialized techniques for first-order partial eigenvalue assignment problem (see Chapter 11 of the book [12]). However, because of several serious computational difficulties, including the inversion of a possible ill-conditioned mass matrix M and the complete loss of the exploitable structures of the matrices, M, K and D , such as the symmetry, sparsity, and definiteness, this approach is not practical. In view of these considerations, in recent years, several techniques for QPEVAP, which work directly in second-order setting and requires the knowledge of only those few eigenvalues and eigenvectors that need to be reassigned, have been developed in (e.g., [9],[15], [13], [14], [17], [18]). Unfortunately, the use of feedback control destroys the symmetry of the updated model. While solution of the QPEVAP does not require that the symmetry is preserved, one of the fundamental requirements of the closely related *Matrix Model Updating Problem*(MMUP)is that the updated model remains symmetric. The MMUP concerns updating of a finite element model in such a way that a set of "unwanted eigenvalues and eigenvectors" from the original model is replaced by suitably given ones and the symmetry of the original model preserved.

The MMUP has been well studied and there exists a large amount of literature on its solution. For an account of the earlier methods, see the authoritative book by Friswell and Mottershead [26]. References to some of the more recent work can be found in[7]. Most of these methods, except the one in [7], are optimization based. The idea is to formulate the problem as a constrained optimization problem and then use some of the existing optimization techniques. In many cases, especially when the problem is solved for an undamped model, explicit solutions can be given (see [26]. The direct method proposed in [7] has the additional feature that it can mathematically guarantee the no spill-over in the updated model. The last paper, however, concerns updating of an undamped model. In [7], a symmetry preserving eigenvalue embedding scheme for a damped model has been proposed and the scheme there is capable of preserving the no spill-over. However, that paper considers assigning only of a given set of eigenvalues but not the eigenvectors. It is to be noted in this context that it is not critically important from practical view point to preserve the no spill-over per say. What is important is to guarantee that spurious modes are not introduced into the frequency range of interests (see [26]).

In this paper, the optimization problem for MMUP for a damped model is formulated in such a way that the well-known "Alternating Projection" technique can be used to our advantage. This has been done by exploiting the special structure offered by the constraint set. Using this technique, a new method for the damped MMUP is proposed. A distinct practical feature of our proposed alternating project method is that it can be implemented using only those few eigenvalues and eigenvectors that are needed to be reassigned. No knowledge of the spectrum and the eigenvectors of $p(\lambda)$ is needed. It is to be noted in this context that most of the eigenvalues and eigenvectors of $p(\lambda)$,in the case when the number of degrees of freedom is quite large, are neither computable using the state-of-the-art computational techniques nor are experimentally measurable. The results of numerical experiments, both on illustrative and benchmark examples, shown that the algorithm is working well.

2 Alternating Projection Method (APM)

In this section we summarize the essentials of APM. The method of alternating projections dates back to John von Neumann [37] who treated the problem of finding the projection of a given point in a Hilbert space H onto the intersection of two closed subspaces: M_1 and M_2. The geometry of APM essentially consists in finding the best approximation to x from $M_1 \cap M_2$, first by projecting x onto M_1, then projecting the obtained result onto M_2, The process can be continued by projecting alternatively onto M_1 and M_2. This way, a sequence of elements is generated which converges to the projection onto the intersection $P_{M_1 \cap M_2} x$ (see figure 1).

The practical usefulness of APM is that, in general, it is easier to compute the projection onto

M_1 and M_2 separately than computing the projection onto $M_1 \cap M_2$. For a complete discussion on alternating projection methods see Deutsch [20], and Escalante and Raydan [24].

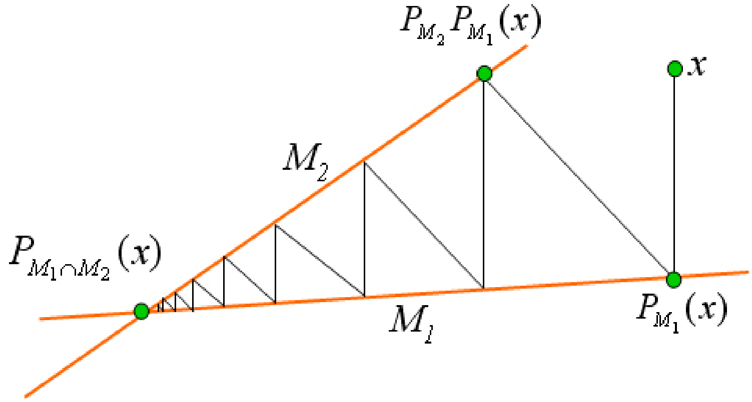

Figure 1: Alternating projections onto the intersection of two closed subspaces.

In 1962, Halperin [28] extended this algorithm to more than two subspaces. Let P_{M_i} $(i = 1, ..., r)$ denote the projection operator onto a linear subspace M_i $(i = 1, ..., r)$ of a Hilbert space H. For the sake of completeness we now present the key theorem associated with APM.

Theorem 1 (Halperin, 1962 [28])) *If M_1,M_2,...,M_r are closed subspaces in H, then for each $x \in H$,*

$$\lim_{n\to\infty} (P_{M_r} P_{M_{r-1}} ... P_{M_1})^n x = P_{\cap_1^r M_i} x. \tag{3}$$

We close this section with some comments. Theorem 1 also holds when, instead of projecting onto subspaces, we project onto linear varieties [20]. Among all extensions and variants of APM, it is worth mentioning that Dykstra and Boyle [21], [5] found a suitable modification of von Neumann's scheme for closed and convex sets. APM and their variants have been used by many researches to solve problems on a wide variety of applications [4, 6, 10, 22, 23, 27, 29, 32, 35, 36, 38].

3 Alternating Projection Approach for MMUP

The problem of interest can be reformulated as an optimization problem as follows: Find matrices $\tilde{D}$ and $\tilde{K}$ such that:

$$\text{Min } \left\| K - \tilde{K} \right\|_F^2 + \left\| D - \tilde{D} \right\|_F^2 \tag{4}$$

subject to:

$$\tilde{K} = \tilde{K}^t, \quad \tilde{D} = \tilde{D}^t$$

$$M(\Lambda_1^*)^2 Y_1 + \tilde{D}(\Lambda_1^*) Y_1 + \tilde{K} Y_1 = 0, \tag{5}$$

where $\Lambda_1^* = diag(\mu_1, \ldots, \mu_p)$ and $Y_1 = \{y_1, \ldots, y_p\}$ are the matrices of the desired eigenvalues and eigenvectors. In finite element model updating literature, these are referred to as "measured" eigenvalue and eigenvector matrices , because in finite element model updating setting, a set of experimentally measured data is needed to be incorporated into an updated finite element model. However, we will simply call them "desired" eigenvalues and eigenvectors, because solution of the MMUP problem is also applicable to controlling dangerous vibration and stabilizing or improving stability in vibration structures.

The norm $\|Z\|_F = (\langle Z, Z\rangle_F)^{\frac{1}{2}}$ is the Frobenius matrix norm of a matrix Z, $\langle Z, Y\rangle_F = tr(Z^T Y)$ is the associated inner product of Z with a matrix Y, and $tr(W)$ denotes the trace of a square matrix W.

As said before, many optimization techniques have been used for solving this Problem. We will apply here APM which fits nicely into our situation since the constraints can be seen as subspaces or linear varieties and the unique solution to the problem lies at their intersection.

For the sake of simplicity, we start by writing (5) as follows:

$$A + \tilde{D}B + \tilde{K}C = 0, \tag{6}$$

where $A = MY_1(\Lambda_1^*)^2$, $B = Y_1(\Lambda_1^*)$, $C = Y_1$, and $A, B, C \in \mathbb{C}^{n\times p}$. We are now ready to write the constrained problem (4) as a function of only one 2×2 block matrix-variable. Indeed, if we define the matrices $X \in \mathbb{R}^{2n\times 2n}$ and $\tilde{X} \in \mathbb{R}^{2n\times 2n}$ as:

$$X = \begin{pmatrix} K & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad \tilde{X} = \begin{pmatrix} \tilde{K} & 0 \\ 0 & \tilde{D} \end{pmatrix},$$

then problem (4) is reduced to the problem of finding the matrix $\tilde{X}$ that solves the following optimization problem:

$$\text{Min} \ \left\|X - \tilde{X}\right\|_F^2 \tag{7}$$

subject to:

$$\tilde{X} = \tilde{X}^t$$

$$A + \tilde{X}_{22}B + \tilde{X}_{11}C = 0.$$

Now we need to write (6) as a function of X. For that we proceed as follows. Let the block matrices W and $\hat{I}$ be defined as

$$\hat{I} = \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix} \qquad \text{and} \qquad W = \begin{pmatrix} C \\ B \end{pmatrix}$$

where $I_{n\times n}$ is the identity matrix of order $n \times n$. Since,

$$\begin{aligned} A + \hat{I}^t * \tilde{X} * W &= A + (I_{n\times n} \quad I_{n\times n}) \begin{pmatrix} \tilde{K} & 0 \\ 0 & \tilde{D} \end{pmatrix} \begin{pmatrix} C \\ B \end{pmatrix} \\ &= A + (\tilde{K} \quad \tilde{D}) \begin{pmatrix} C \\ B \end{pmatrix} \\ &= A + \tilde{K}C + \tilde{D}B, \end{aligned}$$

then $A+\hat{I}^t*\tilde{X}*W = A+\tilde{K}C+\tilde{D}B = A+\tilde{X}_{22}B+\tilde{X}_{11}C$. Therefore, we end up with the following optimization problem:

$$\text{Min } \left\| X - \tilde{X} \right\|_F^2 \tag{8}$$

subject to:

$$\tilde{X} = \tilde{X}^t \tag{9}$$

$$A + \hat{I}^t * \tilde{X} * W = 0. \tag{10}$$

We will find the solution of (8), using APM, projecting back and forth on each one of the two sets associated with the two constraints. The first constraint (9) defines the subspace of symmetric matrices, whose projection is given by

$$P(X) = \frac{X + X^t}{2},$$

i. e., $P(X)$ is the symmetric matrix closest to X (see [25]).

For the second constraint (10), we need to project onto the linear variety

$$V = \left\{ X \in \mathbb{R}^{2n\times 2n} / A + Z^t * X * W = 0 \right\}.$$

The projection onto V can be obtained as a generalization of the standard projection onto a linear variety in the vector space $\mathbb{R}^n$ (i.e., onto a hyperplane). Let us consider for a while the problem of projecting onto the hyperplane

$$h = \{x \in \mathbb{R}^n / a^t x = b\},$$

where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. If $x \in h$ and $y \in h$ then $a^t x = b$ and $a^t y = b$, and so

$$a^t x - a^t y = 0 \Rightarrow a^t(x - y) = 0 \Rightarrow a \perp h.$$

Hence the vector a is orthogonal any vector in h, i. e., a is orthogonal to the hyperplane h. Therefore, the projection onto h is given by

$$P_h(x) = x + \beta a,$$

where β is obtained such that $P_h(x) \in h$. Thus,

$$\beta = \frac{b - a^t x}{a^t a}.$$

Inspired by the projection onto a linear variety in $\mathbb{R}^n$, we proceed as follows to obtain the projection onto V. If $X \in V$ and $Y \in V$ then $A + Z^t * X * W = 0$ and $A + Z^t * Y * W = 0$, and so

$$A + Z^t * X * W - A - Z^t * Y * W = 0,$$

$$Z^t * X * W - Z^t * Y * W = 0,$$

$$Z^t * (X - Y) * W = 0.$$

Therefore the projection of X onto the linear variety V is given by

$$P_V(X) = X + Z\Sigma W^t,$$

where now the matrix Σ plays the role of β in the projection onto h.

Since $P_V(X)$ must be in V, then $A + Z^t * P_V(X) * W = 0$, and from this expression we obtain

$$A + Z^t(X + Z\Sigma W^t)W = 0.$$

Hence,

$$A + Z^t XW + Z^t Z\Sigma W^t W = 0,$$

and so,

$$Z^t Z\Sigma W^t W = -A - Z^t XW.$$

Since $Z^t Z = 2I$, we finally have that

$$W^t W\Sigma^t = -\frac{1}{2}(A^t + W^t X^t Z). \tag{11}$$

Consequently, we obtain the matrix Σ solving a a linear system with multiple right-hand sides where the coefficient matrix $W^t W$ is constant. For numerical reasons it is convenient to use the QR factorization.

In our next result we establish, using the well-known Kolmogorov's criterion [34], that $P_V(X) = X + Z\Sigma W^t$ is the projection of X onto V.

Theorem 2 *If $X \in \mathbb{R}^{2n\times 2n}$ is any given matrix, then the projection onto the linear variety V is given by $P_V(X) = X + Z\Sigma W^t$, where Σ satisfies (11).*

Proof. Let us set $S_0 = P_V(X) = X + Z\Sigma W^t$. We want to demonstrate that $\langle X - S_0, S - S_0\rangle_F \leq 0$ for all $S \in V$ (Kolmogorov's criterion [34]):

$$\langle X - S_0, S - S_0\rangle_F = \langle X - X - Z\Sigma W^t, S - S_0\rangle_F = \langle -Z\Sigma W^t, S - S_0\rangle_F.$$

Hence, using properties of any inner product

$$\langle X - S_0, S - S_0\rangle_F = -\langle \Sigma, Z^t(S - S_0)W\rangle_F = -\langle \Sigma, Z^t SW - Z^t S_0 W\rangle_F.$$

Since $S \in V$ then $Z^t SW = -A$, and so

$$\langle X - S_0, S - S_0\rangle_F = -\langle \Sigma, -A - Z^t S_0 W\rangle_F = \langle \Sigma, A + Z^t S_0 W\rangle_F.$$

Finally, since $S_0 \in V$ then $A + Z^t S_0 W = 0$, and we have that

$$\langle X - S_0, S - S_0\rangle_F = \langle \Sigma, 0\rangle_F = 0 \qquad \blacksquare$$

We are now ready to present our APM algorithm for solving (8):

Algorithm 1 (Updating of the matrices D and K)
Input:

- Real $n \times n$ matrices M, K, D ($M = M^t > 0$, $K = K^t$ and $D = D^t$).
- The diagonal matrix $\Lambda_1^* = diag(\mu_1, \ldots, \mu_p)$ containing the "wanted" eigenvalues.
- The matrix Y_1 = matrix whose columns are the "wanted" eigenvectors $\{y_1, \ldots, y_p\}$.

Step 1: Compute:

$A = MY_1(\Lambda_1^*)^2$

$B = Y_1(\Lambda_1^*)$

$C = Y_1$

Step 2: Form matrix and vectors in blocks, as follows:

$$X = \begin{pmatrix} K & 0 \\ 0 & D \end{pmatrix}, \quad \hat{I} = \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix}, \quad W = \begin{pmatrix} C \\ B \end{pmatrix}$$

Step 3: Compute a QR factorization of W:

$QR = W$

Step 4: **While** $X \notin V$ Do

Step 4.1: Use the QR factorization of W to find Σ
(i.e., solve $R^t R \Sigma^t = -\frac{1}{2}(A^t + W^t X^t Z)$):

$R^t z = -\frac{1}{2}(A^t + W^t X^t Z)$ (Forward substitution)

$R\Sigma^t = z$ (Backward substitution)

Step 4.2: Project X onto V:

$X = X + Z * \Sigma * W^t$

Step 4.3: Project X onto the subspace of symmetric matrices:

$X = (X + X^t)/2$

Output:

- Obtain the updated matrices $\tilde{D}$ and $\tilde{K}$ from the matrix X of Step 4.

Notice that the APM, to find the projection onto the intersection of the subspace of symmetric matrices and V, is included in **Step 4** of Algorithm 1. Therefore, from Theorem 1, Algorithm 1 converges to the unique solution of (8) subject to (9) and (10).

4 Numerical experiments

In this section, we report results on some numerical experiments that illustrate the performance of the new algorithm. In all these experiments, computing was done on a Pentium IV at 3.a GHz with MATLAB 7.0 and 2Mb RAM. The iterative process in Algorithm 1, in all cases, is stopped when

$$\|A + Z^t * X * W\|_F \leq 1.D - 8.$$

Experiment 1: For our first experiment we choose the 4×4 matrices M, D, K, symmetric and positive definite, described in Datta and Sarkissian [17]:

$$M = \begin{pmatrix} 1.4685 & 0.7177 & 0.4757 & 0.4311 \\ 0.7177 & 2.6938 & 1.2660 & 0.9676 \\ 0.4757 & 1.2660 & 2.7061 & 1.3918 \\ 0.4311 & 0.9676 & 1.3918 & 2.1876 \end{pmatrix} \quad D = \begin{pmatrix} 1.3525 & 1.2695 & 0.7967 & 0.8160 \\ 1.2695 & 1.3274 & 0.9144 & 0.7325 \\ 0.7967 & 0.9144 & 0.9456 & 0.8310 \\ 0.8160 & 0.7325 & 0.8310 & 1.1536 \end{pmatrix}$$

$$K = \begin{pmatrix} 1.7824 & 0.0076 & -0.1359 & -0.7290 \\ 0.0076 & 1.0287 & -0.0101 & -0.0493 \\ -0.1359 & -0.0101 & 2.8360 & -0.2564 \\ -0.7290 & -0.0493 & -0.2564 & 1.9130 \end{pmatrix}.$$

The eigenvalues of $P(\lambda) = \lambda^2 M + \lambda D + K$ computed via MATLAB are: $-0.0861 \pm 1.6242i$, $-0.1022 \pm 0.8876i, -0.1748 \pm 1.1922i, -0.4480 \pm 0.2465i$. We want to reassign only the most unstable pair of the eigenvalues; namely, $-0.0861 \pm 1.6242i$ to the locations $-0.1 \pm 1.6242i$. Let the matrix of vectors, to be assigned, be:

$$Y1 = \begin{pmatrix} 1.0000 & 1.0000 \\ 0.0535 + 0.3834i & 0.0535 - 0.3834i \\ 0.5297 + 0.0668i & 0.5297 - 0.0668i \\ 0.6711 + 0.4175i & 0.6711 - 0.4175i \end{pmatrix}.$$

The algorithm calculated the matrices $\tilde{D}$ and $\tilde{K}$ after 113 iterations with $\|D - \tilde{D}\| = 0.9075$ and $\|K - \tilde{K}\| = 2.9507$, where the eigenvalues of $\tilde{P}(\lambda) = \lambda^2 M + \lambda\tilde{D} + \tilde{K}$ are: $-0.1000 \pm 1.6242i, -0.1241 \pm 1.6583i, -0.1605 \pm 1.2195i, -0.3358, -0.6664$, and the columns of $Y1$ are the eigenvectors, corresponding to the eigenvalues $-0.1 \pm 1.6242i$, that is to say, it clearly reassigned the "unwanted" eigenvalues and eigenvectors to the desired sets satisfactorily. form.

Experiment 2: For our next experiment we consider the 30×30 matrices M, D, K, symmetric and positive definite, described in Benner, Laub and Mehrmann [3]. This is a model of a string consisting of coupled springs, dashpots, and masses as shown in Figure 2. The inputs are two forces, one acting on the left end of the string, the other one on the right end.

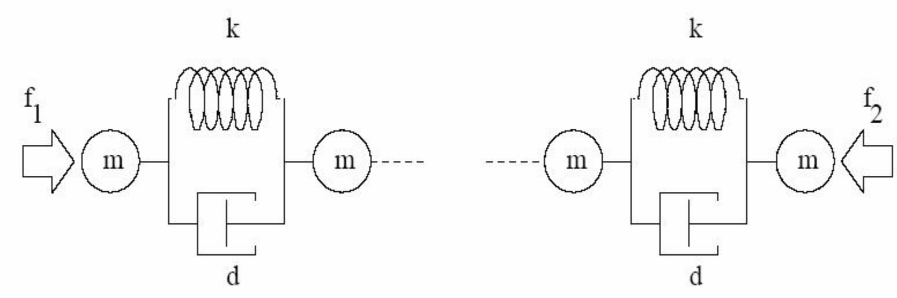

Figure 2: Coupled Spring Experiment

The parameters of this model are, $m = 4.0, d = 4.0, k = 1.0$, and the following matrices are obtained:

$$M = D = 4.0 * I_{30} = \begin{pmatrix} 4 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 4 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 4 \end{pmatrix} \quad K = 1.0 * I_{30} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}.$$

The pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ has 60 eigenvalues, but the eigenvalue that causes the instability is -1.8356e-017, and the rest of the spectrum of $P(\lambda)$ is below -0.0027. We use Algorithm 1 to find $\tilde{D}$ and $\tilde{K}$ such that the "troublesome" eigenvalue was reassigned to -0.018 and the associated eigenvector $Y_1 = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$. After 30 iterations the algorithm solved the problem successfully, where $\|D - \tilde{D}\|_F = 0.0013$, and $\|K - \tilde{K}\|_F = 0.0707$. The rest of the eigenvalues remained below the indicated value.

Experiment 3: For our last experiment we choose the 211×211 matrices M, D, K, symmetric and positive definite, described in Benner, Laub and Mehrmann [3], this example concerns a problem arising in power plants. We consider a model of a rotating axle with several masses placed upon it. These masses may be parts of turbines or generators and are assumed to be symmetric with respect to the axle. The input to the system consists of changing loads which act on the masses. This causes vibrations in the axle. The aim is to minimize the moments between two neighboring masses in order to maximize the life expectancy. Matrices M, D, K are given by

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_{211}) \qquad D = d_{ij}, \text{ where } d_{ij} = \begin{cases} -\gamma_i, & i+1=j; \\ \gamma_{i-1} + \delta_i + \gamma_i, & i=j; \\ -\gamma_j, & i=j+1; \\ 0, & \text{otherwise} \end{cases}$$

$$K = k_{ij}, \text{ where } k_{ij} = \begin{cases} -\kappa_i, & i+1=j; \\ \kappa_{i-1}+\kappa_i, & i=j; \\ -\kappa_j, & i=j+1; \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalue that causes the instability is $1.8281e-007$, and the rest of the 421 of $P(\lambda) = \lambda^2 M + \lambda D + K$ are below -0.2433. To improve the stability of the system this eigenvalue must be changed to -0.016 and the corresponding eigenvector by $Y_1 = \frac{1}{\sqrt{n}}(1,1,\ldots,1)^T$. After 38 iterations the algorithm solved the problem successfully, where $\|D-\tilde{D}\|_F = 0.0374$, and $\|K-\tilde{K}\|_F = 2.3346$. Once again, the rest of the eigenvalues remained below the indicated value.

5 Summary and Conclusion

The MMUP for a second-order system modelling vibrating structures can be formulated as a constrained optimization problem. Several optimization techniques have been proposed in the literature in the past.

In this paper, by exploiting the geometry of the constraint sets, we have developed an alternating projection method to solve MMUP. The feasible region is the intersection of a subspace and a linear variety in the space of matrices. Particularly, we characterized the projection onto the linear variety associated with the pencil that avoids the "unwanted" eigenvalues and eigenvectors. Results of our numerical experiments clearly demonstrate the accuracy of the algorithm. Our future research will be directed towards meeting several other important practical issues related to MMUP. These include the guarantee of the no spill-over property and preservation of the other important physical properties of the model, such as the definiteness and sparsity of the original model. Also finding suitable techniques for acceleration of the speed of convergence of the proposed APM will be studied in details. Regarding the no spill-over property, we have observed in our experiments that the eigenvalues and eigenvectors which have not been reassigned have not moved to undesired locations. However, such observation need to be supported by theoretical results, because as said before, it is not possible to computationally verify this property for large models. Maintaining no spill-over is specially desirable for control of vibration in large structures.

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